COMP 1805 Discrete Structures I
Assignment 3

Due: July 12th 2016, at the end of class

- Write down your name and student number on **every** page. The pages must be **stapled** together.
- You must have a cover page that clearly states **your name, student number, and course number**. If you do not have a cover page with this information, your assignment will not be marked.
- The questions should be answered in order.
- Every part of every question is worth 2 marks. The grading scheme is 2 points for a correct answer, 0 for a completely incorrect answer, and 1 point for something in-between.

Recall that two sets A and B have *the same cardinality* (or size) if there exists a bijection from A to B. A set S is *countable* if (i) it is finite, or (ii) $\mathbb{N}^+$ has the same cardinality as S (i.e., there exists a bijection $f : \mathbb{N}^+ \mapsto S$).

1. Prove that the set $\mathbb{Z}^2 = \mathbb{Z} \times \mathbb{Z}$ is countable.
2. A quadratic equation with integer coefficients has the form
$$ax^2 + bx + c = 0,$$
where a, b, and c are integers (i.e., $a, b, c \in \mathbb{Z}$) and x is a variable. Let QE be the set of all quadratic equations. Prove that the set QE is countable. (Hint: You may use the result of Question 1.)
3. Prove that the set $\mathcal{P}(\mathbb{N}^+)$ (the power set of the positive natural numbers) is uncountable. (Hint: Is i an element of the i-th subset?)
4. Prove that, if a countable set X has the same cardinality as a set Y, then Y is countable.
5. Compute a closed form (without summations) of the following summations:
   (a) $\displaystyle\sum_{i=1}^{3}\sum_{j=1}^{4}\frac{i}{j}$
   (b) $\displaystyle\sum_{i=1}^{n}(10 - i)$
   (c) $\displaystyle\sum_{i=1}^{n}(i^2 + 2 \cdot 3^i)$
   (d) $\displaystyle\sum_{i=1}^{n}\sum_{j=0}^{i}(j + 1)$
   (e) $\displaystyle\sum_{i=1}^{n}\sum_{j=i}^{n}(i^2 - j)$

Recall that, given two functions $f, g : \mathbb{R} \mapsto \mathbb{R}$, we say that f is $O(g(n))$ if

$$\exists c, n_0 \in \mathbb{R}^+ \left(\forall n \geq n_0 \left(f(n) \leq c \cdot g(n)\right)\right).$$

Similarly, we say that f is $\Omega(g(n))$ if

$$\exists c, n_0 \in \mathbb{R}^+ \left(\forall n \geq n_0 \left(f(n) \geq c \cdot g(n)\right)\right).$$

Finally, we say that f is $\Theta(g(n))$ if it is both $O(g(n))$ and $\Omega(g(n))$.

6. Give derivations that prove the following. For convenience, you may assume that the logarithms are in the base of your choice, but you should specify what base you are using in your derivation.
   (a) $(n-7)^2$ is $\Theta(n^2)$.
   (b) $8n^4 - 4n^3 \log n - 6n^2 + 9n \log n$ is $\Omega(n^4)$.
   (c) $2\log(3n^3 - n^2 + 43)$ is $O(\log n)$.
7. In this question, we will prove that $\log(n!) = \Theta(n \log n)$. Recall that $n! = n \times (n-1) \times \cdots \times 2 \times 1$ and $\log(a \times b) = \log a + \log b$.
   (a) Show that $\log(n!)$ is $O(n \log n)$.
   (b) Show that $\log(n!)$ is $\Omega(n \log n)$. (Hint: Consider only the first $n/2$ terms.)